Time-Discretization of Nonlinear Systems with Time Delayed Output via Taylor Series

Zhang Yuanliang, Kil To Chong*
Faculty of Electronics and Information Engineering Chonbuk National University, Duckjin-Dong, Duckjin-Gu, Jeonju 561-756, Korea

An output time delay always exists in practical systems. Analysis of the delay phenomenon in a continuous-time domain is sophisticated. It is appropriate to obtain its corresponding discrete-time model for implementation via a digital computer. A new method for the discretization of nonlinear systems using Taylor series expansion and the zero-order hold assumption is proposed in this paper. This method is applied to the sampled-data representation of a nonlinear system with a constant output time-delay. In particular, the effect of the time-discretization method on key properties of nonlinear control systems, such as equilibrium properties and asymptotic stability, is examined. In addition, 'hybrid' discretization schemes resulting from a combination of the 'scaling and squaring' technique with the Taylor method are also proposed, especially under conditions of very low sampling rates. A performance of the proposed method is evaluated using two nonlinear systems with time-delay output.

Key Words : Output Time-Delay, Scaling and Squaring Technique, Taylor-Series, Time-Discretization

1. Introduction

Time-delays associated with output measurements naturally arise in a variety of engineering applications. Indeed, one may consider cases where the process to be controlled or monitored is located far from the computing unit, the measured output data are transmitted through a low-rate communication system, or of sensor technology that inevitably introduces non-negligible time-delays, which when unaccounted for, may under mine the viability of the process control and monitoring system design. The convergence of communication and computation in control systems and the complex behavior of the control systems with non-negligible time-delays are the two main reasons for the special attention to the time-delayed status. It is difficult to apply the controller design technique developed during the last several score years for finite-dimensional systems to the systems with any time-delays in the variables due to their infinite-dimension. Thus, control system design methods which can solve the systems with time-delays are necessary.

A natural direction for time-delay system control is to attempt to extend the ideas and results of nonlinear non-delay control to systems with delay. Such results include the input-output linearization and decoupling, partial feedback linearization with delay term domination, and extension of control Lyapunov functions (CLF) to delay systems in the form of control Lyapunov-Razumikhin functions (CLRF). Huang et al. (2004), presented a novel start-controlled phase/frequency detector for multiphase-output delay-locked loops. Gudvanden (1997) proposed a sliding Fermat number transform to reduce the input-output delay of finite ring convolvers and correlators. In

* Corresponding Author,
E-mail : kitchong@chonbuk.ac.kr
TEL : +82-63-270-2478; **FAX** : +82-63-270-2475
Faculty of Electronics and Information Engineering Chonbuk National University, Duckjin-Dong, Duckjin-Gu, Jeonju 561-756, Korea. (Manuscript **Received** September 14, 2005; **Revised** April 12, 2006)

many applications magnetic levitation systems are required to have a large operating range. Choi and Baek (2002) applied Time Delay Control (TDC) to a single-axis magnetic levitation system to solve this problem. Germani et al.(2002), presents a new approach for the construction of a state observer for nonlinear systems when the output measurements are available for computations after a non negligible time delay. Lee and Kim (2003) proposed a high level CVT ratio control algorithm to improve engine performance by considering the powertrain response time delay. Cho and Park (2004) proposed a new impedance controller for bilateral teleoperation under a time delay.

Currently, modern nonlinear control strategies are usually implemented on a microcontroller or digital signal processor. As a direct consequence, the control algorithm has to work in discrete-time. For such digital control algorithms, one of the following time discretization approaches is typically used : time-discretization of a continuous time control law designed on the basis of a continuous time system ; and time-discretization of a continuous time system resulting in a discrete-time system and control law design in discrete-time. It is apparent that the second approach is an attractive feature for dealing directly with the issue of sampling. Indeed, the effect of sampling on system-theoretic properties of the continuous-time system is very important because they are associated with the attainment of the design objectives. It should be emphasized that in both design approaches time discretization of either the controller, or the system model is necessary. Furthermore, notice that in the controller design for time-delay systems, the first approach is troublesome due to the infinite-dimensional nature of the underlying system dynamics. As a result the second approach becomes more desirable and will be pursued in the present study.

For digital simulation and design of continuous-time delayed systems, it is often required to have an equivalent discrete-time model available. In the field of the discretization, for the original continuous-time systems with time free case (Franklin et al., 1998), the traditional numerical techniques such as the Euler and Runge-Kutta methods have been used for acquiring sampled-data representations. However, these methods need a small sampling time interval. Due to physical and technical limitations slow sampling is becoming inevitable. A time-discretization method which expands the well-known time-discretization of the linear time-delay system (Franklin et al., 1998 ; Vaccaro, 1995) to a nonlinear continuous-time control system with time-delay (Kazantzis et al., 2003) can solve this problem. And this method is applied to the nonlinear control systems with delayed multi-input (Park et al., 2004a) and the nonlinear control systems with non-affine delayed input (Park et al., 2004b). Scaling and Squaring technique with this time discretization method can also be applied to the nonlinear control systems with delayed multi-input (Zhang and Chong, 2005).

In this paper, the digital state space representation of the dynamic systems with output time-delay is presented. The proposed discretization scheme applies Taylor Series expansion according to the mathematical structure developed for the delay-free nonlinear system (Kazantzis and Kravaris, 1997 ; 1999). In particular, the effect of the time-discretization method on key properties of nonlinear control systems, such as equilibrium properties and asymptotic stability, is examined. Also, the well-known "scaling and squaring" technique, which is widely used to compute the matrix exponential (Higham, 2004) is expanded to the nonlinear case when the sampling period is too large.

Following this introduction, Section 2 briefly describes the basic principles of discretization of nonlinear system with delay-free output. Section 3 presents the detailed discretization of nonlinear system with time delay output which is the main work of this paper, and Section 4 presents the scaling and squaring technique. Section 5 presents the computer simulations of the proposed algorithm. Finally, Section 6 presents a summary and the conclusions drawn from this study.

2. Discretization of Nonlinear System with Delay-Free Output

Initially, output delay-free ($D=0$) nonlinear control systems are considered with a state-space representation of the form:

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + u(t)g(x(t)) \\ y(t) &= h(x(t)) \end{aligned} \tag{1}$$

where $x \in X \subset R^n$ is the vector of states and X is an open and connected set, $u \in R$ is the input variable. It is assumed that $f(x)$ and $g(x)$ are real analytic vector fields on X. The output $y(t) \in R$ is a function of the state x.

An equidistant grid on the time axis with mesh $T = t_{k+1} - t_k$ is considered, where $[t_k, t_{k+1}) = [kT, (k+1)T)$ is the sampling interval and T is the sampling period. It is assumed that for $kT \leq t < kT + T$, the zero-order hold (ZOH) assumption holds true:

$$u(t) = u(kT) \equiv u(k) = \text{constant} \tag{2}$$

Under the ZOH assumption, the solution of (1) is expanded in a uniformly convergent Taylor series (Vydyasagar, 1978):

$$\begin{aligned} x(k+1) &= x(k) + \sum_{l=1}^{\infty} \frac{T^l}{l!} \frac{d^l x}{dt^l}\bigg|_{t^k} \\ &= x(k) + \sum_{l=1}^{\infty} A^{[l]}(x(k), u(k)) \frac{T^l}{l!} \\ y(k) &= h(x(k)) \end{aligned} \tag{3}$$

where $x(k)$ is the value of the state vector x at time $t = t_k = kT$, and $A^{[l]}(x, u)$ are determined recursively by:

$$\begin{aligned} A^{[l]}(x, u) &= f(x) + ug(x) \\ A^{[l+1]}(x, u) &= \frac{\partial A^{[l]}(x, u)}{\partial x}(f(x) + ug(x)), \\ & l = 1, 2, 3, \cdots \end{aligned} \tag{4}$$

The Taylor series expansion of Eq. (3) can offer either an exact sampled-data representation (ESDR) of Eq. (1) by retaining the full infinite series representation of the state vector:

$$\begin{aligned} x(k+1) &= \Phi_T(x(k), u(k)) \\ &= x(k) + \sum_{l=1}^{\infty} A^{[l]}(x(k), u(k)) \frac{T^l}{l!} \\ y(k) &= h(x(k)) \end{aligned} \tag{5}$$

or an approximate sampled-data representation (ASDR) of Eq. (1) resulting from a truncation of the Taylor series of order N:

$$\begin{aligned} x(k+1) &= \Phi_T^N(x(k), u(k)) \\ &= x(k) + \sum_{l=1}^{N} A^{[l]}(x(k), u(k)) \frac{T^l}{l!} \\ y(k) &= h(x(k)) \end{aligned} \tag{6}$$

where the subscript of the map Φ_T^N denotes the dependence on the sampling period T of the sampled-data representation obtained under the above discretization scheme, and the superscript N denotes the finite series truncation order associated with the ASDR of Eq. (6).

Remark 1 It is important to observe that the ESDR of Eq. (5) represents the nonlinear analogue of the exact discretization scheme available for linear systems. Indeed, consider the linear delay-free output control system with a state-space representation of the form:

$$\begin{aligned} \frac{dx(t)}{dt} &= Ax(t) + bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{7}$$

where A, b, C are constant matrices of appropriate dimensions. Integrating Eq. (7) within the sampling interval and under the ZOH assumption results in:

$$\begin{aligned} x(x+1) &= \bar{A}x(k) + \bar{b}u(k) \\ &= \exp(AT)x(k) + \left(\int_0^T \exp(At)\, b\, dt\right)u(k) \\ y(k) &= Cx(k) \end{aligned} \tag{8}$$

where $x(k)$ and $y(k)$ are the value of the state vectors $x(t)$ and $y(t)$ at time $t = t_k = kT$, respectively, $\bar{A} = \exp(AT)$ and $\bar{b} = \int_0^T \exp(At)\, b\, dt$, and the exponential matrix is defined through the uniformly convergent power series:

$$\exp(At) = \sum_{l=0}^{\infty} \frac{A^l t^l}{l!} \tag{9}$$

The notion of the exponential matrix allows a compact expression for the exact sampled-data representation (ESDR). Eq. (8) of the original linear continuous-time system (7). However the underlying series representation of the ESDR Eq. (8) dependence in T emerges, once the definition of the exponential matrix is used. Indeed, Eq. (8) may be rewritten as follows:

$$x(k+1)=\bar{A}x(k)+\bar{b}u(k)$$
$$=x(k)+\sum_{l=1}^{\infty}[A^{l-1}(Ax(k)+bu(k))]\frac{T^l}{l!} \quad (10)$$
$$y(k)=Cx(k)$$

If the recursion formula Eq. (4) is applied to a linear system $f(x)=Ax$, $g(x)=b$, then it is easy to show that $A^{[l]}(x, u)=A^l x+A^{l-1}bu$ and therefore the linear result in Eq. (10) is naturally reproduced.

Definition 1 Given f, an analytic vector field on R^n and h, an analytic scalar field on R^n, the Lie derivative of h with resp
ect to f is defined in local coordinates as (Kazantzis et al., 2003):

$$L_f h(x)=\frac{\partial h}{\partial x_1}f_1+\cdots+\frac{\partial h}{\partial x_n}f_n \quad (11)$$

In light of Definition 1, the solution to the recursive relation (4) may be represented in terms of higher-order Lie derivatives as follows:

$$A_i^{[l]}(x, u)=(L_f+uL_g)^l x_i \quad (12)$$

where the subscript $i=1, \cdots, n$ denotes the i-th component and $L_f=\sum_{i=1}^{n}f_i(x)\frac{\partial}{\partial x_i}$, $L_g=\sum_{i=1}^{n}g_i(x)\frac{\partial}{\partial x_i}$ are Lie derivative operators. This allows for the representation of the series expansion Eq. (3) as a uniformly convergent Lie series for the ESDR:

$$x(k+1)=\Phi_T(x(k), u(k))$$
$$=x(k)+\sum_{l=1}^{\infty}(L_f+uL_g)^l x|_{(x(k),u(k))}\frac{T^l}{l!} \quad (13)$$
$$y(k)=h(x(k))$$

and similarly for the ASDR:

$$x(k+1)=\Phi_T^N(x(k), u(k))$$
$$=x(k)+\sum_{l=1}^{N}(L_f+uL_g)^l x|_{(x(k),u(k))}\frac{T^l}{l!} \quad (14)$$
$$y(k)=h(x(k))$$

3. Time-Discretization of Nonlinear System with Time-Delay Output

A discrete-time nonlinear time-delayed input system can be obtained using Taylor series and it has been shown that the expansion of single dimensional system to n dimensional system is possible. Similarly it can be expanded to output-delay case. The discretization method of general nonlinear system with output delay is developed using Taylor series expansion. The nonlinear continuous control system with output time-delay can be represented as follows.

$$\dot{x}(t)=f(x(t))+u(t)g(x(t))$$
$$y(t)=h(x(t-D)) \quad (15)$$

where $D>0$ is the measurement delay.

Let

$$D=qT+\gamma \quad (16)$$

where $q\in\{0, 1, 2, \cdots\}$ and $0\leq\gamma<T$, i.e., the time-delay D can be represented as an integer multiple of the sampling period plus a fractional part of T (Franklin et al., 1998; Vaccaro, 1995).

Basing on the ZOH assumption and the above notation the sampled-data representation of the nonlinear system with delayed output can be derived from Eq. (3) result in the following equations.

$$u(t)=u(kT-qT-T)\equiv u(k-q-1) \quad \text{if } kT-qT-T\leq t<kT-qT \quad (17)$$

and

$$x(kT-qT-\gamma)=x(k-q-1)+\sum_{l=1}^{\infty}A^l(x(k-q-1), u(k-q-1))\frac{(T-\gamma)^l}{l!} \quad \text{if } kT-qT-T\leq t<kT-qT-\gamma$$
$$x(kT-qT)=x(kT-qT-\gamma)+\sum_{l=1}^{\infty}A^l(x(kT-qT-\gamma), u(k-q-1))\frac{\gamma^l}{l!} \quad \text{if } kT-qT-\gamma\leq t<kT-qT \quad (18)$$
$$y(kT)=h(x(kT-D))=h(x(kT-qT-r))$$

where $x(k)$ and $A^{[l]}(x, u)$ are the same as above delay-free case.

Theorem 1 Let x^0 be an equilibrium point of the original nonlinear continuous-time system: $\frac{dx(t)}{dt}=f(x)+ug(x)$, $y(t)=h(x(t))$, that belongs to the continuous-time equilibrium manifold: $E^c=\{x\in R^n \mid \exists u\in R: f(x)+ug(x)=0\}$, and $u=u^0$ be the corresponding equilibrium value of the input variables: $f(x^0)+u^0g(x^0)=0$. The x^0 belongs to the discrete-time equilibrium manifold: $E^d=\{x\in R^n \mid \exists u\in R: \Phi_T^D(x, u)=x\}$ of the ESDR: $x(k+1)=\Phi_T^D(x(k), u(k))$ and ASDR: $x(k+1)=\Phi_T^{N,D}(x(k), u(k))$ obtained under the proposed Taylor-Lie discretization method, with $u=u^0$ being the corresponding equilibrium values of the input variables: $\Phi_T^D(x^0, u^0)$ and $\Phi_T^{N,D}(x^0, u^0)=x^0$.

Proof: x^0 is the equilibrium point and u^0 is the corresponding equilibrium values of the input variable.

$$\Rightarrow A^{[1]}(x^0, u^0)=f(x^0)+u^0g(x^0)=0$$

$$\Rightarrow A^{[l+1]}(x^0, u^0)=\frac{\partial A^{[l]}(x^0, u^0)}{\partial x}A^{[1]}(x^0, u^0)=0$$

for all $l\in\{1, 2, 3, \cdots\}$

$$\Rightarrow \Phi_{T-\gamma}(x^0, u^0)=x^0+\sum_{l=1}^{\infty}A^{[l]}(x^0, u^0)\frac{(T-\gamma)^l}{l!}=x^0$$

$$\Rightarrow \Phi_T^D(x^0, u^0)=\Phi_\gamma(\Phi_{T-\gamma}(x^0, u^0), u^0)=x^0$$

Similar arguments apply to the $\Phi_T^{N,D}$ map of the ASDR. Therefore, x_0 belongs to the discrete-time equilibrium manifold E^d of the ESDR and ASDR for any finite truncation order N.

Theorem 1 essentially states that the output time-delay nonlinear control system equilibrium properties are preserved under the proposed Taylor discretization method.

Theorem 2 Assume that matrix $M=\left[\frac{\partial f}{\partial x}+u^0\frac{\partial g}{\partial x}\right](x^0)$ is Hurwitz, so that x^0 is a locally asymptotically stable equilibrium point of the delay-free system:

$$\frac{dx(t)}{dt}=f(x(t))+g(x(t))u(t),\ y(t)=h(x(t))$$

Then:

(1) x^0 is a locally asymptotically stable equilibrium point of the ESDR.

(2) x^0 is a locally asymptotically stable equilibrium point of the ASDR for sufficiently large N, when T is fixed.

The following technical lemma is essential and its proof can be found in (Kazantzis and Kravaris, 1997).

Lemma 1 In the single input status, let x^0 be an equilibrium point of $\frac{dx(t)}{dt}=f(x(t))+g(x(t))u(t-D)$ that corresponds to $u=u^0$. For any analytic scalar field $h(x)$ and positive integer l the following equality holds:

$$\begin{aligned}&\frac{\partial}{\partial x}[L_f+uL_g]^l h(x)|_{(x^0,\ u^0)}\\&=\frac{\partial h}{\partial x}\left[\frac{\partial f}{\partial x}+u^0\frac{\partial g}{\partial x}\right]^l(x^0)\end{aligned}\tag{19}$$

The i-th row of the matrix $\frac{\partial \Phi_T^N}{\partial x}(x^0, u^0)$ can be calculated as follows:

$$\begin{aligned}\frac{\partial \Phi_{i,T}^N}{\partial x}(x^0, u^0)&=\sum_{l=0}^{N}\frac{\partial}{\partial x}[(L_f+uL_g)^l x_i]|_{(x^0,u^0)}\frac{T^l}{l!}\\&=\sum_{l=0}^{N}\frac{\partial x_i}{\partial x}\left(\frac{\partial f}{\partial x}+u^0\frac{\partial g}{\partial x}\right)^l(x^0)\frac{T^l}{l!}\end{aligned}\tag{20}$$

Proof: Due to Lemma 1.

$$\begin{aligned}\frac{\partial \Phi_T^N}{\partial x}(x^0, u^0)&=\sum_{l=0}^{N}\left(\frac{\partial f}{\partial x}+u^0\frac{\partial g}{\partial x}\right)^l(x^0)\frac{T^l}{l!}\\&=\sum_{l=0}^{N}M^l\frac{T^l}{l!}\end{aligned}\tag{21}$$

for an ASDR of finite truncation order N, or

$$\begin{aligned}\frac{\partial \Phi_T}{\partial x}(x^0, u^0)&=\exp\left[\left(\frac{\partial f}{\partial x}+u^0\frac{\partial g}{\partial x}\right)(x^0)\,T\right]\\&=\exp(MT)\end{aligned}\tag{22}$$

for the ESDR $(N\rightarrow\infty)$

Now consider the ESDR with time-delay D. Note that:

$$\begin{aligned}\frac{\partial \Phi_T^D}{\partial x}(x^0, u^0)&=\frac{\partial \Phi_\gamma}{\partial x}(\Phi_{T-\gamma}(x^0, u^0), u^0)\\&=\exp(M\gamma)\exp(M(T-\gamma))\\&=\exp(MT)\end{aligned}\tag{23}$$

Since M is Hurwitz, it can be inferred that all the eigenvalues of $\frac{\partial \Phi_T^D}{\partial x}(x^0, u^0)$ have modulus less than one, and hence x^0 is a locally asymptotically stable equilibrium point of the ESDR.

Consider now the ASDR. One obtains:

$$\frac{\partial \Phi_T^{N,D}}{\partial x}(x^0, u^0) = \frac{\partial \Phi_\gamma^N}{\partial x}(\Phi_{T-\gamma}^N(x^0, u^0), u^0) = \sum_{l_1=0}^{N}\sum_{l_2=0}^{N} M^{l_1+l_2}\frac{\gamma^{l_1}(T-\gamma)^{l_2}}{l_1!\,l_2!} \tag{24}$$

Now notice that for a stable eigenvalue λ_i of $M(\mathrm{Re}[\lambda_i]<0)$, the corresponding eigenvalue a_i of $\frac{\partial \Phi_T^{N,D}}{\partial x}(x^0, u^0)$: $a_i = \sum_{l_1=0}^{N}\sum_{l_2=0}^{N}\lambda^{l_1+l_2}\frac{\gamma^{l_1}(T-\gamma)^{l_2}}{l_1!\,l_2!}$ is stable only when $|a_i|<1$.

Since for a fixed T and as $N\to\infty$, $a_i\to\exp(\lambda_i\gamma)\exp(\lambda_i(T-\gamma))=\exp(\lambda_i T)$, one can always find a sufficiently large order of truncation N such that: $|a_i|<1$.

4. Scaling and Squaring Technique (SST)

When T is considerably large $A^{[l]}T^l/l!$ might become extremely large before it becomes small at higher powers, where convergence takes over. An SST which is also called extrapolation to the limit technique in the numerical analysis literature can be applied to solve this kind of problem. By applying SST when T is large enough, one can divide the interval (t, t_{k+1}) into 2^m equally spaced subintervals and use a small Taylor expansion of order N with a time step $T/2^m$ for the 2^m intermediate subintervals to substitute the larger order N' used in the single-step Taylor method case.

Assume now that $\Omega(N', T): R^n\to R^n$ is the operator that corresponds to the Taylor expansion of order N' with a time step T, and when it acts on $x(kT)$ the outcome is:

$$x(kT+T)=\Omega(N', T)x(kT) \tag{25}$$

where $\Omega(N', T)(\cdot)=I+\sum_{l=1}^{N'}A^{[l]}(x(k), u(k))\frac{T^l}{l!}$.

Using operator notation, the resulting discrete-time system may now be written as follows:

$$x(kT+T)=\left[\Omega\left(N, \frac{T}{2^m}\right)\right]^{2m}x(kT) \tag{26}$$

How to choose the parameters of N and m is an important implementation. Different values of N and m can reflect different requirements of the discretization performance. In this paper we use the two elements: i) simplicity and computing time; and ii) numerical convergence and accuracy requirements to select these two kinds of parameters. In fact the criterion for selecting an appropriate m involves a comparison of the magnitude of the sampling period T with the fastest time constant $1/\rho$ of the original continuous-time system. If T is small compared to $2/\rho$, then we can set $m=0$ and we apply the single-step Taylor-Lie series method. When T is larger than the fastest time constant $2/\rho$, we apply the SST. The sampling interval is subdivided into 2^m subintervals, and a low-order N single step Taylor discretization method is applied for each subinterval. Thus, this method indicates that the following inequality should hold: $T/2^m<2/\rho$.

And the SST can be applied to the nonlinear control systems with time-delayed output. In this case, we do not consider the single sampling interval T but the subintervals of $T-\gamma$ and γ. The method to choose m can also be used by changing T of that preceding equality into these subintervals of $T-\gamma$ and γ. That is

$$m_{T-\gamma}=\max\left(\left[\log_2\left(\frac{T-\gamma}{\theta}\right)\right]+1,\ 0\right) \text{ and } \quad m_\gamma=\max\left(\left[\log_2\left(\frac{\gamma}{\theta}\right)\right]+1,\ 0\right) \tag{27}$$

5. Simulation

Two examples are considered in the computer simulations. The system 1 is a simple nonlinear system with delayed output measurements (Germani et al., 2002):

$$\begin{aligned}\dot{x}_1(t)&=c_1x_2(t)\\ \dot{x}_2(t)&=c_2x_1(t)+c_3x_1(t)x_2(t)+c_4x_1(t)u(t)\\ y(t)&=c_5x_1(t-D)\end{aligned} \tag{28}$$

with all $c_i\neq 0$. In this example we assume $c_1=1$, $c_2=-2$, $c_3=1$, $c_4=1$, $c_5=1.5$, $u(t)=1$, $x_1(0)=1$ and $x_2(0)=-1$. There are three cases considered in system1. The parameters used in the simulation are; $T=0.001$s, $D=0.0028$s for case 1:

T=0.005s, D=0.008s for case 2 and T=0.01s, D=0.006s for case 3.

This system can be discretized as followings;

$$
\begin{aligned}
x_1(kT-D) &= x_1(kT-qT-T)+x_2(kT-qT-T)\,d \\
&+0.5\begin{bmatrix} -2x_1(kT-qT-T)+x_1(kT-qT-T)x_2(kT-qT-T) \\ +x_1(kT-qT-T)\,u(k-q-1) \end{bmatrix} d^2 \\
x_2(kT-D) &= x_2(kT-qT-T) \\
&+\begin{bmatrix} -2x_1(kT-qT-T)+x_1(kT-qT-T)x_2(kT-qT-T) \\ +x_1(kT-qT-T)\,u(k-q-1) \end{bmatrix} d \\
&+0.5\begin{bmatrix} (-2+x_2(kT-qT-T)+u(k-q-1))\,x_2(kT-qT-T) \\ +x_1(kT-qT-T)\,(-2x_1(kT-qT-T) \\ +x_1(kT-qT-T)x_2(kT-qT-T)+x_1(kT-qT-T)\,u(k-q-1)) \end{bmatrix} d^2 \\
y(k) &= 1.5x_1(kT-D)
\end{aligned} \qquad (29)
$$

The simulations have been performed by Maple. During the simulations process the Taylor coefficient was determined as N=2. Table 1 and Fig. 1 show the attributes of the simulation for case 1; Table 2 and Fig. 2 are for case 2; and Table 3 and Fig. 3 similarly show the simulation results of case 3. The absolute value of the output error for case 1 ranges from 0.0825×10^{-3} to 5.3993×10^{-3}; case 2 ranges from 0.0098×10^{-2} to 1.0180×10^{-2}; and case 3 ranges from 0.0114×10^{-2} to 1.0929×10^{-2}. From these results we can observe that the discretization scheme is more accurate if the sampling period is smaller. That is as the sampling time increases in size, you have to use an increasingly large Taylor order N to achieve the better results.

The simulation of system 2 has been performed

Table 1 Numerical result of output for case 1 of System 1; T=0.001, D=0.0028

	$X1(t-D)$		$Y1(t)$	
	MATLAB	Maple	MATLAB	Maple
500	0.2798	0.2787	0.4197	0.4181
1000	−0.6100	−0.6109	−0.9150	−0.9164
1500	−1.1593	−1.1596	−1.7389	−1.7395
2000	−1.2626	−1.2625	−1.8939	−1.8938
2500	−1.0868	−1.0865	−1.6302	−1.6298
3000	−0.7751	−0.7747	−1.1626	−1.1620
3500	−0.4024	−0.4020	−0.6037	−0.6030
4000	−0.0063	−0.0059	−0.0095	−0.0089
4500	0.3901	0.3905	0.5852	0.5857
5000	0.7640	0.7642	1.1460	1.1464

Table 2 Numerical result of output for case 2 of System 1; T=0.005, D=0.008

	$X1(t-D)$		$Y1(t)$	
	MATLAB	Maple	MATLAB	Maple
100	0.2891	0.2873	0.4337	0.4310
200	−0.6019	−0.6034	−0.9028	−0.9052
300	−1.1561	−1.1567	−1.7341	−1.7350
400	−1.2633	−1.2632	−1.8949	−1.8947
500	−1.0895	−1.0890	−1.6343	−1.6335
600	−0.7787	−0.7780	−1.1681	−1.1670
700	−0.4024	−0.4057	−0.6037	−0.6086
800	−0.0105	−0.0098	−0.0157	−0.0146
900	0.3861	0.3867	0.5791	0.5801
1000	0.7603	0.7609	1.1405	1.1413

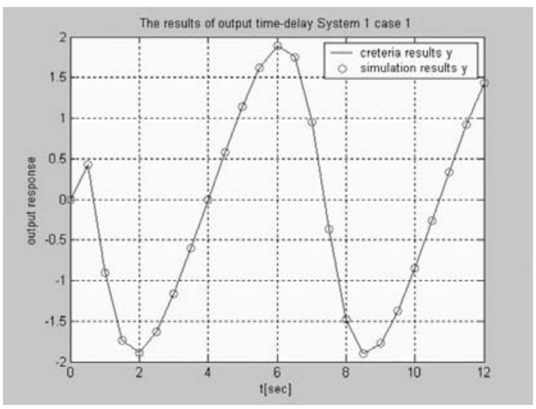

Fig. 1 The response of output for case 1 of System 1

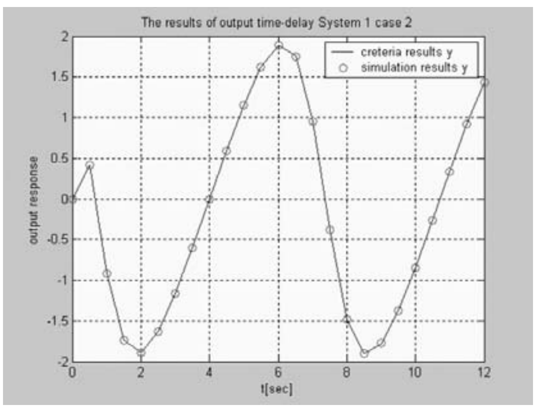

Fig. 2 The response of output for case 2 of System 1

for the following nonlinear continuous system:

$$\begin{aligned} \dot{x}(t) &= -3x(t) + u(t) - u(t)x(t) - x^2(t) \\ y(t) &= 2.5\exp(ax(t-D)) \\ x(0) &= 0 \\ u(t) &= 0.9(-1)^k \end{aligned} \tag{30}$$

This system can be discretized as the following (Taylor order $N=2$):

$$\begin{aligned} x(kT-D) &= x(k-q-1) \\ &+ \begin{bmatrix} -3x(k-q-1)+u(k-q-1) \\ -x(k-q-1)u(k-q-1)-x(k-q-1)^2 \end{bmatrix} d \\ &+ 0.5[-3-u(k-q-1)-2x(k-q-1)] \\ &\begin{bmatrix} -3x(k-q-1)+u(k-q-1) \\ -x(k-q-1)u(k-q-1)-x(k-q-1)^2 \end{bmatrix} d^2 \end{aligned} \tag{31}$$

Table 3 Numerical result of output for case 3 of System 1; $T=0.01$, $D=0.006$

	$X1(t-D)$		$Y1(t)$	
	MATLAB	Maple	MATLAB	Maple
50	0.2855	0.2819	0.4283	0.4229
100	−0.6050	−0.6082	−0.9075	−0.9122
150	−1.1573	−1.1586	−1.7360	−1.7378
200	−1.2630	−1.2628	−1.8946	−1.8941
250	−1.0885	−1.0874	−1.6327	−1.6311
300	−0.7773	−0.7759	−1.1660	−1.1639
350	−0.4049	−0.4034	−0.6074	−0.6051
400	−0.0089	−0.0073	−0.0133	−0.01102
450	0.3876	0.3891	0.5814	0.5836
500	0.7617	0.7630	1.1426	1.1445

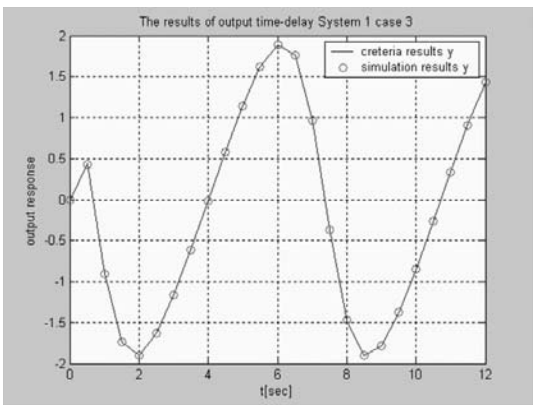

Fig. 3 The response of output for case 3 of System 1

The parameters selected for the first simulation in system 2 are; $a=20$, $T=0.05$s, $D=0.02$s. The Taylor series orders are chosen as $N=1$, 2, 3 for three cases. Table 4 shows the simulation results in which the exact values are obtained using Matlab. Figure 4 shows the error of the output. For the second simulation of the system 2, we choose parameter are as $a=2$, $T=0.5$s, $D=0.2$s, and the Taylor order $N=3$, 5, 9. Table 5 and Fig. 5 present the simulation results of the second case of the system 2. These preceding results reveal that it is more accurate if the Taylor order N is larger.

In the last simulation, the parameter have been determined as $a=1$, $T=5$s, and $D=2$s. First, a single-step Taylor method has been worked out.

Table 4 Numerical result of output for case 1 of System 2

Time Step	Y			
	exact	Taylor ($N=1$)	Taylor ($N=2$)	Taylor ($N=3$)
1	4.1611	4.2900	4.1566	4.1611
2	3.1891	3.2503	3.1872	3.1891
3	3.7429	3.8976	3.7374	3.7430
4	2.9096	2.9927	2.9067	2.9096
5	3.4613	3.6346	3.4551	3.4614
6	2.7190	2.8179	2.7154	2.7190
7	3.2669	3.4542	3.2601	3.2670
8	2.5861	2.6970	2.5821	2.5861
9	3.1302	3.3283	3.1230	3.1302
10	2.4919	2.6122	2.4876	2.4919

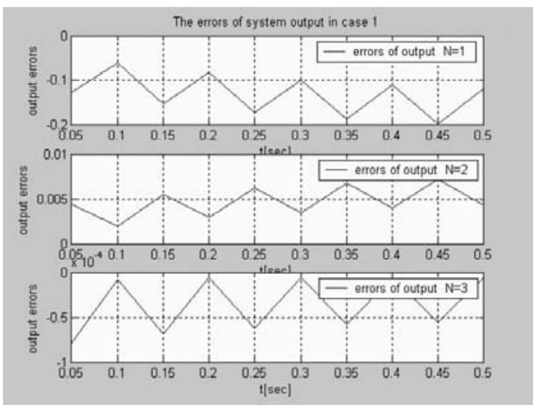

Fig. 4 The response of output for case 1 of System 2

Table 5 Numerical result of output for case 2 of System 2

Time Step	Y exact	Taylor (N=3)	Taylor (N=5)	Taylor (N=9)
1	3.4207	3.4869	3.4143	3.4207
2	2.0493	2.0018	2.0412	2.0492
3	2.9968	2.9949	2.9776	2.9969
4	1.9887	1.9427	1.9799	1.9886
5	2.9768	2.9713	2.9571	2.9769
6	1.9856	1.9396	1.9767	1.9856
7	2.9758	2.9700	2.9560	2.9759
8	1.9855	1.9394	1.9766	1.9854
9	2.9757	2.9699	2.9560	2.9758
10	1.9855	1.9394	1.9766	1.9854

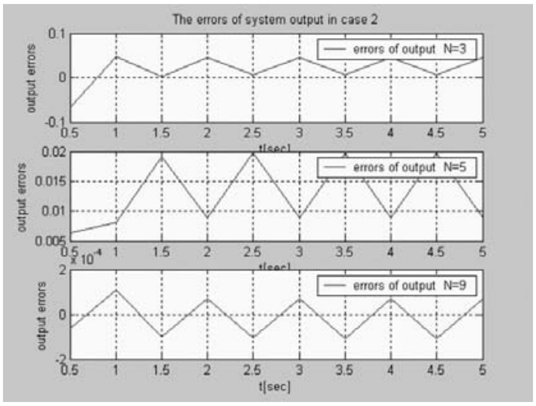

Fig. 5 The response of output for case 2 of System 2

The attributes of the simulation are presented in Table 6. As shown in Table 6, it is difficult to discretize the system accurately if the sampling period is very large. It is recommended to implement the SST in this case. The simulation results of the SST implementation are shown in Table 7. The required computational time of this SST simulation are shown in Table 8.

Table 6 Numerical result of output for case 3 of System 2

Time Step	Y exact	Taylor (N=3)	Taylor (N=7)	Taylor (N=3)
1	0.2185	41.21	−3349.93	*
2	−0.5702	*	*	*
3	0.2185	*	*	*
4	−0.5702	*	*	*
5	0.2185	*	*	*
Time Step	**Y exact**	**Taylor (N=1)**	**Taylor (N=2)**	**Taylor (N=3)**
1	3.1106	*	*	*
2	1.4135	*	*	*
3	3.1106	*	*	*
4	1.4135	*	*	*
5	3.1106	*	*	*

*Denotes order of magnitude greater than 10^5

Table 7 Numerical results of SST for case 3 of system 2

Time Step	Y exact	Taylor (N=1) L=6, M=6	Taylor (N=2) L=5, M=5	Taylor (N=3) L=4, M=4	Taylor (N=4) L=4, M=4	Taylor (N=5) L=3, M=3
1	3.1106	3.1106	3.1106	3.1106	3.1106	3.1106
2	1.4135	1.4099	1.4140	1.4132	1.4135	1.4129
3	3.1106	3.1106	3.1106	3.1106	3.1106	3.1106
4	1.4135	1.4099	1.4140	1.4132	1.4135	1.4129
5	3.1106	3.1106	3.1106	3.1106	3.1106	3.1106
6	1.4135	1.4099	1.4140	1.4132	1.4135	1.4129
7	3.1106	3.1106	3.1106	3.1106	3.1106	3.1106
8	1.4135	1.4099	1.4140	1.4132	1.4135	1.4129
9	3.1106	3.1106	3.1106	3.1106	3.1106	3.1106
10	1.4135	1.4099	1.4140	1.4132	1.4135	1.4129

Table 8 Required computing time for SST in case 2 of system 2 (sec)

N	1	1	1	2	2	2	3	3	3
l	6	7	8	5	6	7	4	5	6
m	6	7	8	5	6	7	4	5	6
t	57.2	135.8	328.6	45.3	104.2	240.6	30.6	67.8	149.3
N	4	4	4	5	5	5	6	7	8
l	4	5	6	3	4	5	3	3	3
m	4	5	6	3	4	5	3	3	3
t	40.2	87.2	194.3	24.9	55.3	118.7	32.3	41.1	54.4

The simulation has been performed for 3000 steps.

6. Conclusions

This paper has presented an approach for the discrete-time representation of a nonlinear control system with output time-delay in control. This system is based on the ZOH assumption and the Taylor-Series expansion, which is obtained as a solution of continuous-time systems. The mathematical structure of the new discretization scheme is explored and characterized as useful for establishing concrete connections between numerical and system-theoretic properties. In particular, the effect of the time-discretization method on key properties of nonlinear control systems with output time-delay, such as equilibrium properties and asymptotic stability, is examined. Also, the well known "scaling and squaring" technique is expanded to the nonlinear case when the sampling period is too large. The proposed scheme provides a finite-dimensional representation for nonlinear systems with output time-delay enabling existing controller design techniques to be applied to them.

The performance of the proposed discretization scheme is evaluated using two nonlinear systems. For these two nonlinear control systems various sampling rates and time-delay values are considered, demonstrating the accuracy of the proposed discretization scheme.

Acknowledgments

The authors are grateful for the support provided by a grant from the Korean Institute of Industrial Technology Evaluation and Planning (R12-1998-026-02003-0) and Korean Electric Power Company (R-2004-B-120).

References

Cho, H. C. and Park, J. H., 2004, "Design and Stability Analysis of Impedance Controller for Bilateral Teleoperation under a Time Delay," *KSME International Journal*, Vol. 18, No. 7, pp. 1131～1139.

Choi, J. S. and Baek, Y. S., 2002, "A Single DOF Magnetic Levitation System using Time Delay Control and Reduced-Order Observer," *KSME International Journal*, Vol. 16, No. 12, pp. 1643～1651.

Franklin, G. F., Powell, J. D. and Workman, M. L., 1998, *Digital Control of Dynamic Systems*. Addison-Wesley, New York.

Germani, A., Manes, C. and Pepe, P., 2002, "A New Approach to State Observation of Nonlinear Systems with Delayed Output," *Automatic Control, IEEE Transactions*, Vol. 47, Issue 1, pp. 96～101.

Gudvanden, S., 1997, "A Class of Sliding Fermat Number Transforms that Admit a Tradeoff Between Complexity and Input-Output Delay," *IEEE Transactions on Signal Processing*, Vol. 45, No. 12, pp. 3094～3096.

Higham, N. J., 2004, "The Scaling and Squaring Method for the Matrix Exponential Revisited," *Numerical Analysis Report 452*, Manchester Center for Computational Mathematics.

Huang, P. J., Chen, H. M. and Chang, R. C., 2004, "A Novel Start-Controlled Phase/Frequency Detector for Multiphase-Output Delay-Locked Loops," *2004 IEEE Asia-Pacific Conference on Advanced System Integrated Circuits* (*AP-ASIC2004*), pp. 68～71.

Kazantzis, N. and Kravaris, C., 1997, "System-Theoretic Properties of Sampled-Data Representations of Nonlinear Systems Obtained via Taylor-Lie Series," *Int. J. of Control*, Vol. 67, pp. 997～1020.

Kazantzis, N. and Kravaris, C., 1999, "Time-Discretization of Nonlinear Control Systems via Taylor Methods," *Comp. Chem. Engn.*, Vol. 23, pp. 763～784.

Kazantzis, N., Chong, K. T., Park, J. H. and Parlos, A. G., 2003, "Control-Relevant Discretization of Nonlinear Systems with Time-Delay Using Taylor-Lie Series," *American Control Conference*, pp. 149～154.

Lee, H. and Kim, H., 2003, "CVT Ratio Control for Improvement of Fuel Economy by Considering Powertrain Response Lag," *KSME International Journal*, Vol. 17, No. 11, pp. 1725～1731.

Park, J. H., Chong, K. T., Kazantzis, N. and Parlos, A. G., 2004a, "Time-Discretization of Nonlinear Systems with Delayed Multi-Input using Taylor Series," *KSME International Journal*, Vol. 18, No. 7, pp. 1107～1120.

Park, J. H., Chong, K. T., Kazantzis, N. and Parlos, A. G., 2004b, "Time-Discretization of Non-Affine Nonlinear Systems with Delayed Input using Taylor-Series," *KSME International Journal*, Vol. 18, No. 8, pp. 1297～1305.

Vaccaro, R. J., 1995, *Digital Control*. McGraw-Hill, New York.

Vydyasagar, M., 1978, *Nonlinear Systems Analysis*, Prentice Hall, Englewood Cliffs, New York.

Zhang, Y. and Chong, K. T., 2005, "Discretization of Nonlinear Systems with Delayed Multi-Input via Taylor Series and Scaling and Squaring Technique," *KSME International Journal*, Vol. 19, No. 11, pp. 1975～1987.